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Viscous Fluid Universe Interacting with Scalar Field

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Taking a spherically symmetric isotropic line element, the case of a viscous fluid distribution interacting with scalar field is investigated. Four new solutions are obtained and the models are found to be expanding ones. Their physical and geometrical properties are studied.

1. INTRODUCTION

It is well known that meson particles with the charge of the electron and masses of the order of magnitude of 200 electron masses are found in cosmic rays. These particles have a good deal to do with the nulcear forces. The scalar meson field is a matter field and is associated with zero-spin chargeless particles such as π and κ mesons. The study of such a field in general relativity has been initiated to provide an understanding of the nature of space-time and the gravitational field associated with neutral elementary particles of zero spin. Scalar fields, as they help in explaining the creation of matter in cosmological theories, represent matter fields with spinless quanta and can describe the gravitational fields. Yukawa (1935) introduced the short-range meson field. Yukawa's theory is based on the assumption that all interactions must be transmitted through space from point to point by the mediation of a field, which is consistent with the principle of relativity; that is, the equations must be Lorentz-invariant.

Subsequently many authors took interest in the study of scalar fields. For example, Das (1962), Hyde (1963), and Das and Agarwal (1974) obtained solutions for the coupled gravitational and scalar fields. Rao et al. (1976) studied the interaction of a massive scalar field with a perfect fluid for the conformally flat, spherically symmetric metric. Banerjee and Santosh (1981), Froyland (1982), and Accioly et al. (1984) obtained different

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solutions for Einstein's field equations taking consideration of scalar fields. On the other hand, various authors (e.g., Heller et al., 1973; Heller and Klimek, 1975; Lukaćs, 1981; Maiti, 1982) studied the importance of viscous fluid from the cosmological solution point of view.

Here the motivation for taking the scalar field in addition to the viscous fluid as energy-momentum tensor is with a view to obtaining solutions for the cosmological model and to study its physical properties. It is noted that all the normal stresses are equal due to the spherical symmetry assumed and the shear viscosity factor drops from the field equations. The bulk viscosity need not be zero for the viscous fluid distribution interacting with the scalar field. The coefficient of bulk viscosity ζ in the process of studying the solutions is found to be accompanied by a change in volume (that is, in density).

2. FIELD EQUATIONS

For this problem the line element considered is

$$ds^2 = e^{\gamma}\, dt^2 - e^{\beta}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) \tag{1}$$

where β is a function of r and t, and γ is a function of t only.

The energy-momentum tensor for a viscous fluid interacting with a massive scalar field is given by

$$T_{\mu\nu} = R_{\mu\nu} + S_{\mu\nu} \tag{2}$$

where $R_{\mu\nu}$ and $S_{\mu\nu}$ are, respectively, the energy-momentum tensors for the viscous fluid and the massive scalar field.

Here,

$$R_{\mu\nu} = \rho u_\mu u_\nu + (p - \zeta\theta) H_{\mu\nu} - 2\eta\sigma_{\mu\nu} \tag{3}$$

where p is the isotropic pressure, ρ is the fluid density, ζ and η are the coefficients of bulk and shear viscosity, $\theta = u^{\mu}_{;\mu}$ is the expansion factor of the fluid lines, $H_{\mu\nu}$ is the projection tensor defined by

$$H_{\mu\nu} = u_\mu u_\nu - g_{\mu\nu}$$

$\sigma_{\mu\nu}$ is the shear tensor given by

$$\sigma_{\mu\nu} = \tfrac{1}{2}(u_{\mu;\tau} H^{\tau}_{\nu} + u_{\nu;\tau} H^{\tau}_{\mu}) - \tfrac{1}{3}\theta H_{\mu\nu}$$

and u_μ is the flow vector satisfying the relation

$$g^{\mu\nu} u_\mu u_\nu = 1 \tag{4}$$

In addition,

$$S_{\mu\nu} = \varphi_\mu \varphi_\nu - \tfrac{1}{2} g_{\mu\nu}(\varphi_\alpha \varphi^\alpha - M^2 \varphi^2) \tag{5}$$

where the scalar potential $\varphi = \varphi(r, t)$ satisfies the Klein–Gordon equation

$$g^{\mu\nu}\varphi_{;\mu\nu} + M^2\varphi = \varepsilon \tag{6}$$

Here, $\varepsilon = \varepsilon(r, t)$ is the source density of the scalar field and M is related to the mass of the zero-spin particle by $M = m/\hbar$ ($\hbar = h/2\pi$, where h is Planck's constant).

Considering the comoving coordinate system, we get

$$u^1 = u^2 = u^3 = 0, \qquad u^4 = e^{-\gamma/2} \tag{7}$$

We note that the orthogonality conditions for viscous fluid are satisfied here identically, namely

$$\begin{aligned} H_{\mu\nu}u^{\nu} &= 0 \\ \sigma_{\mu\nu}u^{\nu} &= 0 \\ \omega_{\mu\nu}u^{\nu} &= 0 \\ \dot{u}_{\nu}u^{\nu} &= 0 \end{aligned} \tag{8}$$

where $\dot{u}_{\nu} = u_{\nu;\mu}H^{\mu}$ are acceleration components, and $\omega_{\mu\nu}$ are rotation tensors given by

$$\omega_{\mu\nu} = \tfrac{1}{2}(u_{\mu;\tau}H^{\tau}_{\nu} - u_{\nu;\tau}H^{\tau}_{\mu})$$

Thus for the line element (1) the Einstein field equations given by

$$G^{\mu}_{\nu} = -8\pi G T^{\mu}_{\nu}$$

are

$$-e^{-\beta}\left(\frac{\beta'^2}{4} + \frac{\beta'}{r}\right) + e^{-\gamma}\left(\ddot{\beta} + \tfrac{3}{4}\dot{\beta}^2 - \frac{\dot{\beta}\dot{\gamma}}{2}\right)$$
$$= -8\pi G(p - \zeta\theta) - 4\pi G(e^{-\beta}\phi'^2 + e^{-\gamma}\dot{\varphi}^2 - M^2\varphi^2) \tag{9}$$

$$-e^{-\beta}\left(\frac{\beta''}{2} + \frac{\beta'}{2r}\right) + e^{-\gamma}\left(\ddot{\beta} + \frac{3}{4}\dot{\beta}^2 - \frac{\dot{\beta}\dot{\gamma}}{2}\right)$$
$$= -8\pi G(p - \zeta\theta) + 4\pi G(e^{-\beta}\varphi'^2 - e^{-\gamma}\dot{\varphi}^2 + M^2\varphi^2) \tag{10}$$

$$-e^{-\beta}\left(\beta'' + \frac{1}{4}\beta'^2 + \frac{2\beta'}{r}\right) + \tfrac{3}{4}e^{-\gamma}\dot{\beta}^2$$
$$= 8\pi G\rho + 4\pi G(e^{-\beta}\phi'^2 + e^{-\gamma}\dot{\varphi}^2 + M^2\varphi^2) \tag{11}$$

$$-\dot{\beta}' = 8\pi G\varphi'\dot{\varphi} \tag{12}$$

In this paper an overdot and a prime denote partial differentiation with respect to t and with respect to r, respectively, and a semicolon followed by a subscript denotes covariant differentiation.

3. SOLUTIONS OF THE FIELD EQUATIONS

3.1. Case I

In this case the number of unknowns to be determined is greater than the number of equations at hand. Therefore, we need more equations for solving the unknowns. Thus we assume some relations and try to solve the field equations.

Subtracting (10) from (9), we have

$$\frac{\beta''}{2}-\frac{\beta'}{2r}-\frac{\beta'^2}{4}=-8\pi G\varphi'^2 \tag{13}$$

3.1.1. Case I(A)

In this case we take

$$16\pi G\varphi'^2=\beta'^2 \tag{14}$$

Then from (13) we get

$$\frac{\beta''}{2}+\frac{\beta'^2}{4}-\frac{\beta'}{2r}=0$$

which gives

$$\beta=2\log(r^2/4-k)-m \tag{15}$$

where m is an arbitrary constant and k is an arbitrary function of time. Thus, from (14) we get

$$\varphi'=\frac{r}{(\pi G)^{1/2}(r^2-4k)} \tag{16}$$

Now from (12) and (16) we have

$$\dot{\varphi}=-\frac{2\dot{k}}{(\pi G)^{1/2}(r^2-4k)} \tag{17}$$

Then (16) and (17) give

$$\varphi=\frac{1}{2(\pi G)^{1/2}}\log(r^2-4k) \tag{18}$$

From (9) and (10) we get

$$16\pi Gp-16\pi G\zeta\theta+8\pi Ge^{-\gamma}\dot{\varphi}^2-8\pi GM^2\varphi^2$$
$$=e^{-\beta}\left(\frac{\beta'^2}{4}+\frac{\beta''}{2}+\frac{3\beta'}{2r}\right)-2e^{-\gamma}\left(\ddot{\beta}+\frac{3}{4}\dot{\beta}^2-\frac{\dot{\beta}\dot{\gamma}}{2}\right)$$

which gives

$$\begin{aligned}p=\frac{1}{2\pi G}\{&16e^{m}(r^2-4k)^{-3}+e^{-\gamma}(r^2-4k)^{-2}(2r^2\ddot{k}+4k\dot{k}\dot{\gamma}\\&-8k\ddot{k}-20\dot{k}^2-\dot{k}\dot{\gamma}r^2)-24\pi G\zeta\dot{k}e^{-\gamma/2}(r^2-4k)^{-1}\\&-4e^{-\gamma}\dot{k}^2(r^2-4k)^{-2}+\tfrac{1}{4}M^2[\log(r^2-4k)]^2\}\end{aligned}\tag{19}$$

Again from (11) we get

$$\rho=\frac{1}{\pi G}\left\{e^{-\gamma}\dot{k}^2(r^2-4k)^{-2}-256e^{m}(r^2-4k)(r^2-6k)-\frac{M^2}{8}[\log(r^2-4k)]^2\right\}\tag{20}$$

Also from (6) we have

$$e^{-\beta}\left[\varphi''+\left(\frac{2}{r}+\frac{\beta'}{2}\right)\varphi'\right]-e^{-\gamma}\left[\ddot{\varphi}+\left(\frac{3}{2}\dot{\beta}-\frac{\dot{\gamma}}{2}\right)\dot{\varphi}\right]+M^2\varphi=\varepsilon$$

which gives

$$\begin{aligned}\varepsilon=(\pi G)^{-1/2}[&48(r^2-4k)^{-3}\exp(\gamma+m)-16\dot{k}^2(r^2-4k)^{-2}\\&+(2\ddot{k}-\dot{\gamma}\dot{k})(r^2-4k)^{-1}+\tfrac{1}{2}M^2\log(r^2-4k)]\end{aligned}\tag{21}$$

3.1.2. Case I(B)

In this case we assume

$$8\pi G\varphi'^2=\beta'/2r\tag{22}$$

Then from (13) we get

$$\beta''/2-\beta'^2/4=0$$

which gives

$$\beta=g-2\log(c-r/2)\tag{23}$$

where c is an arbitrary constant and g is an arbitrary function of time. Thus from (22) and (23) we get

$$\varphi=\frac{2}{(8\pi G)^{1/2}}\sin^{-1}\left(\frac{r}{2c}\right)^{1/2}+c_1\tag{24}$$

where c_1 is an arbitrary constant.

Now from (9) and (10) we have

$$p=\frac{1}{16\pi G}\left\{24\pi G\zeta\dot{g}e^{-\gamma/2}+8\pi GM^2\left[\frac{2}{(8\pi G)^{1/2}}\sin^{-1}\left(\frac{r}{2c}\right)^{1/2}+c_1\right]^2\right\}$$

$$+\frac{3c}{2r}e^{-g}+2e^{-\gamma}(\tfrac{1}{2}\dot{g}\dot{\gamma}-\ddot{g}-\tfrac{3}{4}\dot{g}^2)-\tfrac{1}{4}e^{-g} \tag{25}$$

From (11) we get

$$\rho=\frac{1}{8\pi G}\left\{\frac{3}{8}e^{-g}+\frac{3}{4}e^{-\gamma}\dot{g}^2-\frac{9c}{4r}e^{-g}\right.$$

$$\left.-4\pi GM^2\left[2(8\pi G)^{-1/2}\sin^{-1}\left(\frac{r}{2c}\right)^{1/2}+c_1\right]^2\right\} \tag{26}$$

Also from (6) we have

$$\varepsilon=\tfrac{3}{4}c(8\pi G)^{-1/2}e^{-g}r^{-3/2}(2c-r)^{1/2}+M^2\left[2(8\pi G)^{-1/2}\sin^{-1}\left(\frac{r}{2c}\right)^{1/2}+c_1\right] \tag{27}$$

3.2. Case II

Here we take $M=0$ and $\varepsilon=0$. In this case the field equations (9)-(12) become

$$-e^{-\beta}\left(\frac{\beta'^2}{4}+\frac{\beta'}{r}\right)+e^{-\gamma}\left(\ddot{\beta}+\frac{3}{4}\dot{\beta}^2-\frac{\dot{\beta}\dot{\gamma}}{2}\right)$$

$$=-8\pi G(p-\zeta\theta)-4\pi G(e^{-\beta}\varphi'^2+e^{-\gamma}\dot{\varphi}^2) \tag{28}$$

$$-e^{-\beta}\left(\frac{\beta''}{2}+\frac{\beta'}{2r}\right)+e^{-\gamma}\left(\ddot{\beta}+\frac{3}{4}\dot{\beta}^2-\frac{\dot{\beta}\dot{\gamma}}{2}\right)$$

$$=-8\pi G(p-\zeta\theta)+4\pi G(e^{-\beta}\varphi'^2-e^{-\gamma}\dot{\varphi}^2) \tag{29}$$

$$-e^{-\beta}\left(\beta''+\frac{1}{4}\beta'^2+\frac{2\beta'}{r}\right)+\tfrac{3}{4}e^{-\gamma}\dot{\beta}^2$$

$$=8\pi G\rho+4\pi G(e^{-\beta}\varphi'^2+e^{-\gamma}\dot{\varphi}^2) \tag{30}$$

and

$$-\dot{\beta}'=8\pi G\varphi'\dot{\varphi} \tag{31}$$

We also have two conservation equations given by

$$T^{\mu}_{\nu;\mu}=0$$

namely,

$$\frac{\partial}{\partial r}(-p+\tfrac{3}{2}\zeta\dot{\beta}e^{-\gamma/2}-\tfrac{1}{2}e^{-\gamma}\dot{\varphi}^2)-\left(3\beta'+\frac{4}{r}\right)(-p+\tfrac{3}{2}\zeta\dot{\beta}e^{-\gamma/2}-\tfrac{1}{2}e^{-\gamma}\dot{\varphi}^2)=0 \tag{32}$$

and

$$\dot{\rho}+\tfrac{3}{2}(p+\rho)\dot{\beta}+e^{-\beta}\varphi'\dot{\varphi}'+e^{-\gamma}\dot{\varphi}\ddot{\varphi}-\tfrac{1}{2}e^{-\gamma}\dot{\varphi}^2\dot{\gamma}+\tfrac{3}{2}e^{-\gamma}\dot{\beta}\dot{\varphi}^2-\tfrac{9}{4}\zeta e^{-\gamma/2}\dot{\beta}^2=0 \qquad (33)$$

The Klein-Gordon equation becomes in this case

$$g^{\mu\nu}\varphi_{;\mu\nu}=0$$

which gives

$$e^{-\beta}\left[\varphi''+\left(\frac{2}{r}+\frac{\beta'}{2}\right)\varphi'\right]-e^{-\gamma}\left[\ddot{\varphi}+\left(\frac{3}{2}\dot{\beta}-\frac{\dot{\gamma}}{2}\right)\dot{\varphi}\right]=0 \qquad (34)$$

3.2.1. Case II(A)

In this case we assume

$$\dot{\varphi}=0 \qquad (35)$$

Then from (34) we get

$$\varphi''+(2/r+\beta'/2)\varphi'=0$$

which gives

$$\varphi'=Yr^{-2}e^{-\beta/2} \qquad (36)$$

where Y is an arbitrary function of time.

Here, we take the case

$$Y=-t \qquad (37)$$

Thus from (36) we get

$$\varphi'=-tr^{-2}e^{-\beta/2} \qquad (38)$$

From (28) and (29) we have

$$e^{-\beta}\left(\frac{\beta''}{2}+\frac{\beta'}{2r}-\frac{\beta'^2}{4}-\frac{\beta'}{r}\right)=8\pi G\,e^{-\beta}\varphi'^2$$

that is,

$$\frac{\beta''}{2}-\frac{\beta'^2}{4}-\frac{\beta'}{2r}=8\pi Gr^{-4}t^2\,e^{-\beta} \qquad (39)$$

[using (38)]. A solution of this equation is

$$\beta=\log 8\pi G+2\log t-2\log r \qquad (40)$$

From (38) and (40) we get

$$\varphi=a-(8\pi G)^{-1/2}\log r \qquad (41)$$

where a is an arbitrary constant.

Again from (28) and (29) we have

$$16\pi G(p-\zeta\theta)=e^{-\beta}\left(\frac{\beta''}{2}+\frac{\beta'^2}{4}+\frac{3\beta'}{2r}\right)-2e^{-\gamma}\left(\ddot{\beta}+\frac{3}{4}\dot{\beta}^2-\frac{\dot{\beta}\dot{\gamma}}{2}\right)$$

which gives

$$16\pi Gp=48\pi G\zeta t^{-1}e^{-\gamma/2}+\frac{1}{8\pi G}t^{-2}+2e^{-\gamma}(t^{-2}-t^{-1}\dot{\gamma}) \tag{42}$$

Also from (30) we get

$$8\pi G\rho=\frac{1}{16\pi G}t^{-2}+3t^{-2}e^{-\gamma} \tag{43}$$

Now using (42) and (43) in (33), we have

$$e^{-\gamma}t^{-1}-e^{-\gamma}\dot{\gamma}+t^{-1}/24\pi G=0$$

the solution of which is

$$\gamma=-\log\left(bt^{-1}-\frac{1}{24\pi G}\right) \tag{44}$$

where b is an arbitrary constant.

Thus, from (42) we get

$$p=3\zeta t^{-1}\left(bt^{-1}-\frac{1}{24\pi G}\right)^{1/2}+\frac{1}{24\pi G}t^{-2} \tag{45}$$

Also from (43) we have

$$\rho=\frac{3b}{8\pi G}t^{-3}-\frac{1}{128\pi^2G^2}t^{-2} \tag{46}$$

3.2.2. Case II(B)

In this case we take

$$\varphi'=0 \tag{47}$$

Then from (34) we get

$$\ddot{\varphi}+(\tfrac{3}{2}\dot{\beta}-\tfrac{1}{2}\dot{\gamma})\dot{\varphi}=0 \tag{48}$$

From (9) and (10) we have

$$\frac{\beta''}{2}-\frac{\beta'^2}{4}-\frac{\beta'}{2r}=0$$

A solution of this equation is

$$\beta=\log z-4\log r \tag{49}$$

where z is an arbitrary function of time.

Now from (48) and (49) we get

$$\dot{\varphi} = -fe^{\gamma/2}z^{-3/2}r^{6} \tag{50}$$

where f is an arbitrary function of r. Since φ is independent of r [by our assumption (47)], therefore in view of (50) we take

$$f = c_0 r^{-6} \tag{51}$$

where c_0 is an arbitrary constant.

Then

$$\dot{\varphi} = -c_0 e^{\gamma/2}z^{-3/2} \tag{52}$$

Again adding up (28) and (29), we get

$$16\pi Gp - 16\pi G\zeta\theta + 8\pi Ge^{-\gamma}\dot{\varphi}^2$$
$$= e^{-\beta}\left(\frac{\beta'^2}{4}+\frac{\beta''}{2}+\frac{3\beta'}{2r}\right) - 2e^{-\gamma}\left(\ddot{\beta}+\frac{3}{4}\dot{\beta}^2-\frac{\dot{\beta}\dot{\gamma}}{2}\right)$$

which gives

$$8\pi Gp = 12\pi G\zeta\left(\frac{\dot{z}}{z}\right) e^{-\gamma/2} - 4\pi Gc_0^2 z^{-3}$$
$$-e^{-\gamma}\left(\frac{\ddot{z}}{z}-\frac{1}{4}\frac{\dot{z}^2}{z^2}-\frac{\dot{z}}{2z}\dot{\gamma}\right) \tag{53}$$

Also from (30) we get

$$8\pi G\rho = \tfrac{3}{4}e^{-\gamma}(\dot{z}/z)^2 - 4\pi Gc_0^2 z^{-3} \tag{54}$$

Now using (53) and (54) in (32), we have

$$\left(3\beta'+\frac{4}{r}\right)e^{-\gamma}\left(\frac{\ddot{z}}{z}-\frac{\dot{z}^2}{4z^2}-\frac{\dot{z}}{2z}\dot{\gamma}\right) = 0$$

that is,

$$-8r^{-1}e^{-\gamma}\left(\frac{\ddot{z}}{z}-\frac{\dot{z}^2}{4z^2}-\frac{\dot{z}}{2z}\dot{\gamma}\right) = 0 \tag{55}$$

Therefore,

$$\frac{\ddot{z}}{z}-\frac{\dot{z}^2}{4z^2}-\frac{\dot{z}}{2z}\dot{\gamma} = 0$$

the solution of which is

$$\gamma = \log(cz^{-1/2}\dot{z}^2) \tag{56}$$

where c is a constant of integration.

Now from (53) and (56) we get

$$p=\tfrac{3}{2}\zeta c^{-1/2}z^{-3/4}-\tfrac{1}{2}c_0^2z^{-3} \tag{57}$$

Again from (54) and (56) we have

$$\rho=\frac{3}{32\pi Gc}z^{-3/2}-\tfrac{1}{2}c_0^2z^{-3} \tag{58}$$

Here (52) and (56) give

$$\varphi=\tfrac{4}{3}c_0c^{1/2}z^{-3/4}+c_1 \tag{59}$$

where c_1 is an arbitrary constant.

4. CONCLUSIONS

4.1. Case I(A)

In this case the line element takes the form

$$ds^2=e^\gamma\,dt^2-e^{-m}(\tfrac{1}{4}r^2-k)^2(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\varphi^2) \tag{60}$$

where m is an arbitrary constant and γ and k are arbitrary functions of time.

Here the fluid density and the pressure both are found to be decreasing functions of r.

Now, for this model to be a realistic one we must have the following restrictions:

(i) $\rho>0$
(ii) $p\geq 0$
(iii) $\rho\geq p$

which respectively give

$$\begin{aligned}&e^{-\gamma}\dot{k}^2+3\cdot 2^9ke^m(r^2-4k)^3\\&\quad>2^8e^mr^2(r^2-4k)^3+\tfrac{1}{8}M^2(r^2-4k)^2[\log(r^2-4k)]^2\end{aligned} \tag{61}$$

$$\begin{aligned}&3\cdot 2^5\pi G\zeta\dot{k}e^{-\gamma/2}+2e^{-\gamma}(r^2-4k)^{-1}(8k\ddot{k}+28\dot{k}^2+\dot{k}\dot{\gamma}r^2)\\&\quad\leq 2^6e^m(r^2-4k)^{-2}+M^2(r^2-4k)[\log(r^2-4k)]^2\\&\qquad+2^3e^{-\gamma}(r^2\ddot{k}+k\dot{k}\dot{\gamma})(r^2-4k)^{-1}\end{aligned} \tag{62}$$

and

$$\begin{aligned}&8e^{-\gamma}\dot{k}^2+12\pi G\zeta\dot{k}(r^2-4k)e^{-\gamma/2}+\tfrac{1}{4}e^{-\gamma}(8k\ddot{k}+\dot{k}\dot{\gamma}r^2)\\&\quad\geq 2^8e^m(r^2-4k)^3(r^2-6k)+e^{-\gamma}(r^2\ddot{k}+k\dot{k}\dot{\gamma})\\&\qquad+8e^m(r^2-4k)^{-1}+\tfrac{1}{4}M^2(r^2-4k)^2[\log(r^2-4k)]^2\end{aligned} \tag{63}$$

The "expansion factor" θ of the fluid lines is given by

$$\theta=\frac{12\dot{k}}{4k-r^2}e^{-\gamma/2} \tag{64}$$

We see that the source density of the scalar field is a decreasing function of r and the scalar potential φ is a decreasing function of time.

The rotation tensors ω_{ij} and the shear σ all come out to be zero.

For this model the spectral shift in wavelength, as measured at the origin, will be

$$(\lambda+\delta\lambda)/\lambda = b_1 e^{-\gamma/2} \tag{65}$$

where b_1 is an arbitrary constant.

4.2. Case I(B)

In this case the metric takes the form

$$ds^2 = e^{\gamma}\,dt^2 - (c-\tfrac{1}{2}r)^{-2}e^{g}(dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2) \tag{66}$$

where c is an arbitrary constant and g and γ are arbitrary functions of time.

Here the fluid density and the pressure both are found to be decreasing with the increase of radial distance, but not appreciably.

For this model to be a realistic one we must have the restrictions

$$\begin{aligned} &3r(e^{-g}+2e^{-\gamma}\dot{g}^2) \\ &\quad > 32\pi GM^2 r[2(8\pi G)^{-1/2}\sin^{-1}(r/2c)^{1/2}+c_1]^2 + 18ce^{-g} \end{aligned} \tag{67}$$

$$\begin{aligned} &48\pi G\zeta\dot{g}e^{-\gamma/2} + 3cr^{-1}e^{-g} + 2e^{-\gamma}\dot{\gamma}\dot{g} \\ &\quad + 16\pi GM^2[2(8\pi G)^{-1/2}\sin^{-1}(r/2c)^{1/2}+c_1]^2 \\ &\quad \geq 4e^{-\gamma}\ddot{g} + 3e^{-\gamma}\dot{g}^2 + \tfrac{1}{2}e^{-g} \end{aligned} \tag{68}$$

and

$$\begin{aligned} &e^{-g} + 3e^{-\gamma}\dot{g}^2 + 2e^{-\gamma}\ddot{g} \\ &\quad \geq 24\pi G\zeta\dot{g}e^{-\gamma/2} + 6cr^{-1}e^{-g} + e^{-\gamma}\dot{\gamma}\dot{g} \\ &\qquad + 16\pi GM^2[2(8\pi G)^{-1/2}\sin^{-1}(r/2c)^{1/2}+c_1]^2 \end{aligned} \tag{69}$$

The expansion factor of the fluid lines is given by

$$\theta = \tfrac{3}{2}e^{-\gamma/2}\dot{g} \tag{70}$$

Here also, the source density of the scalar field is a decreasing function of r.

For this model, the spectral shift will be

$$(\lambda+\delta\lambda)/\lambda = c_2 e^{-\gamma/2} \tag{71}$$

where c_2 is an arbitrary constant.

4.3. Case II(A)

In this case the line element comes out to be

$$ds^2=\left(\frac{24\pi Gt}{24\pi Gb-t}\right)dt^2-8\pi Gt^2r^{-2}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\varphi^2) \quad (72)$$

Here the pressure and the fluid density are both decreasing functions of time.

The temporal history of the model does not span the entire time period $0<t<\infty$, since it is restricted by the reality conditions involving the density and the pressure. For a realistic distribution we must have

$$48\pi Gb>t \quad (73)$$

$$0<t\leq 24\pi Gb \quad (74)$$

and

$$\frac{3b}{8\pi G}t^{-3}\geq\frac{1}{8\pi G}\left(\frac{1}{3}+\frac{1}{16\pi G}\right)t^{-2}+3\zeta t^{-1}\left(bt^{-1}-\frac{1}{24\pi G}\right)^{1/2} \quad (75)$$

From (73)-(75) we obtain the limits within which t must lie as

$$0<t<24\pi Gb \quad (76)$$

In this model the scalar field comes out to be a function of r only and it decreases with the increase of r.

Here the rotation tensors ω_{ij} are identically zero. The shear σ also happens to be zero.

The "expansion factor" θ of the fluid lines is given by

$$\theta=3t^{-1}\left(bt^{-1}-\frac{1}{24\pi G}\right)^{1/2} \quad (77)$$

Since the expansion factor is positive here, we see that our model universe in this case is an expanding one.

For this model, the spectral shift in wavelength, as measured at the origin, will be

$$\frac{\lambda+\delta\lambda}{\lambda}=b_1\left(bt^{-1}-\frac{1}{24\pi G}\right)^{1/2} \quad (78)$$

where b and b_1 are arbitrary constants.

4.4. Case II(B)

In this case the line element takes the form

$$ds^2=cz^{-1/2}\dot{z}^2\,dt^2-zr^{-4}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\varphi^2) \quad (79)$$

where c is an arbitrary constant and z is an arbitrary function of time.

Here the pressure and the fluid density both decrease with time, but not appreciably.

For the distribution to be a realistic one we must have

$$z > (16\pi G c_0^2 c/3)^{2/3} \tag{80}$$

$$z \geq (c_0^2 c^{1/2}/3\zeta)^{4/9} \tag{81}$$

$$z \leq (c^{-1/2}/16\pi G\zeta)^{4/3} \tag{82}$$

Thus the limitations for the temporal history of the model are given by

$$(\tfrac{16}{3}\pi G c_0^2 c)^{2/3} < z \leq \left(\frac{c^{-1/2}}{16\pi G\zeta}\right)^{4/3} \tag{83}$$

In this model the scalar field comes out to be a function of time only and it decreases with the increase of time.

The rotation tensors ω_{ij} and the shear σ are identically zero.

The "expansion factor" θ is given by

$$\theta = \tfrac{3}{2} c^{-1/2} z^{-3/4} \tag{84}$$

Since the expansion factor is positive, we see that our model universe in this case is an expanding one.

For this model the spectral shift will be

$$(\lambda + \delta\lambda)/\lambda = c_2 z^{1/4} (\dot{z})^{-1} \tag{85}$$

where c_2 is an arbitrary constant and z is an arbitrary function of time.

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